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The Non-abelian Specker-Group Is Free

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The paper investigates a group H which can be constructed as a subgroup of the inverse limit of the finitely generated free groups F_n by taking only those elements $g \in \varprojlim(F_n)$, for which there exists some bound $b(g)$ such that none of the generators of any of the groups F_n occurs more often than $b(g)$ times in any of the entries of at least one sequence of F_n-elements describing g. By using a similar condition, H can also be described as a subgroup of the fundamental group of the Hawaiian Earrings. Despite not having any obvious candidate for a free basis, H is proven to be free by a non-constructive basis selection method. Since H is related to $\varprojlim(F_n)$ the same way as the classical Specker group is related to $\mathbb{Z}^{\mathbb{N}}$, we decided to call H the "non-abelian Specker group."

Key Words: word sequences, inconstructive basis selection of a free group, fundamental group of the Hawaiian Earrings.

In 1950 Specker published a paper where he gave a proof that (in modern language) $\mathbb{Z}^{\mathbb{N}}$, i.e., the additive group of all sequences of positive or negative integers, is not free abelian and that the corresponding subgroup of those sequences which only have finitely many different entries is free abelian modulo the continuum hypothesis [Sp, Satz VI, and Einleitung]. The latter result was considered to be the most interesting of Specker's paper, so that in modern literature the corresponding group of bounded integer-sequences is consequently called the "Specker group" or, since there have been generalizations, the "classical Specker group."

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Specker's proof, since it used the continuum hypothesis, could not offer a concrete basis for this group, and one should not expect that such a basis ever could be named, since the variety of patterns (how finitely many integers could be placed on the infinitely many positions of a sequence) is too large (at least, it is uncountable). In addition, the result of Felgner and Schulz, that if the axiom of choice was replaced by one of its alternatives, the classical Specker group could be proven to be not free abelian, implies that any basis construction that deserves the attribute "concrete" is impossible, cf. [FS, Satz 2.4]. By the way, proofs that the classical Specker group is free abelian are now known that avoid using the continuum hypothesis. The first such proof was offered by E. Nöbeling 18 years after Specker's paper [Noe, Satz 1]. In the current paper we want to show that the situation in the non-abelian case is analogous: By imposing a similar boundedness condition we get a group which by a non-constructive method for choosing a basis can be proven to be free, although a concrete basis cannot be named, nor is there any obvious candidate for such a basis.

What can be used as a "non-abelian analogue" to $\mathbb{Z}^{\mathbb{N}}$ or to the Specker-subgroup? In the corresponding finitely generated context the free group of n non-commuting symbols is regarded as the non-abelian analogue to $\mathbb{Z}^n$. However, since concatenating two infinite words may result in a word which is "infinite in its interior," we see that any concept for describing the non-abelian analogue to $\mathbb{Z}^{\mathbb{N}}$ must be sophisticated enough to in principle describe such arrangements of infinitely many non-commuting letters which are infinite in the interior also. By considering homomorphic images of words of such kind in those subgroups which are only generated by finitely many of our symbols, one obtains finite words again, and hence considering sequences of such finite words where the predecessor is a homomorphic image of its successor in a simpler group may be understood as a suitable concept to in principle describing those kinds of infinite words we need here. Hence we will consider $\varprojlim(F_n)$ with $F_n = \langle \alpha_1, \dots, \alpha_n \mid - \rangle$ as a non-abelian analogue of $\mathbb{Z}^{\mathbb{N}}$, where this inverse-limit-construction is based on mapping F_n to F_{n-1} by trivializing α_n and on behaving as the identity on the remaining α_i-symbols. Note that the subgroup of $\varprojlim(F_n)$, where for each group-element g there exist individual bounds $b_i(g)$ for the number of occurrences for each of the α_i, is sometimes separately discussed, since this group describes the fundamental group of the Hawaiian Earrings (cf. Fig. 1). The latter group (which we denote by $\pi_1(Y)$) and $\varprojlim(F_n)$ are known to be not free; for a proof see [Higm, Proof of Theorem 6; dSm, Sec. 3; Zas, 4.8]. The group mainly discussed in this paper is a subgroup of $\pi_1(Y)$ which is defined by the fact that for each group-element g there has to exist a bound $b_\infty(g)$ which restricts the number of occurrences of each of the symbols α_i in any

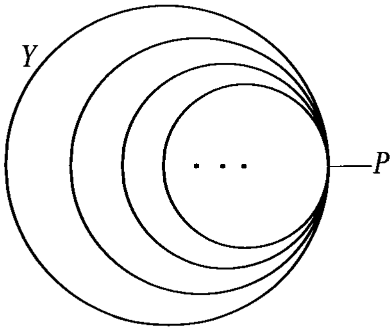


FIG. 1. A topological space Y which is called "The Hawaiian Earrings." It consists of a countable union of circle segments with radii given by a null-sequence. It is essential that these circles all have one tangent point in common, but are disjoint otherwise. Y is topologized as a subset of the Euclidean plane.

of the words of the $\varprojlim(F_n)$-sequence describing g. By using a phrase which has been introduced by the author in [Zas, 2.3] as an alternative concept for describing $\pi_1(Y)$, we will call this group "the group of bounded word sequences," and we will consider it as the non-abelian analogue to the Specker group.

Hence we can rephrase the main result of this paper as follows:

0.1 THEOREM. *The group of bounded word sequences is free.*

The symbol H is reserved to denote this group. Other free subgroups of $\pi_1(Y)$ can be found in a preprint of Cannon and Conner [C–C]. An alternative proof of Theorem 0.1 and a generalization are offered in a preprint of K. Eda [Eda] which appeared after the first submission of this paper.

0.2 *Remark.* In the remainder of this paper the expression $b_\infty(g)$ is used to denote the smallest bound satisfying the conditions described above. Hence for each $g \in \varprojlim(F_n)$ the index $b_\infty(g)$ is either a uniquely defined natural or the symbol ∞. Note that $b_\infty(g) < \infty$ if and only if $g \in H$.

1. ON THE CONCEPT OF WORD SEQUENCES

Word sequences have been introduced by the author in a preliminary version [Zas] as a self-contained combinatorial concept for describing the fundamental group of the Hawaiian Earrings. In the published version of [Zas] they are for the sake of brevity introduced only as an alternative notation for a subgroup of $\varprojlim(F_n)$, namely, for precisely the subgroup that by [Gr1, Gr2, MM] is isomorphic to the fundamental group of the Hawaiian Earrings. Since the proofs in the current paper will rely on the combinatorial structure of word sequences to a far greater extent than in [Zas], this paper will give the complete definition of word sequences as a self-contained combinatorial concept. However, for proving that they describe the fundamental group of the Hawaiian Earrings we will refer to [Gr1, Gr2, MM] as far as possible.

1.0 Basic convention. Let $\mathbb{A} = \{\alpha_1, \alpha_2, \alpha_3, \dots\}$ be a countable alphabet which is implicitly used wherever in this paper word sequences are considered.

1.1 Definition. A *word sequence* $\omega = (\omega_i)_{i \in \mathbb{N}}$ is a sequence of words, where the ith word ω_i is a finite string of letters taken from the set $\{\alpha_1, \alpha_1^{-1}, \alpha_2, \alpha_2^{-1}, \dots, \alpha_i, \alpha_i^{-1}\}$ such that ω_{i-1} is obtained from ω_i by eliminating all letters which are either α_i or α_i^{-1} (cf. the first sentence in Section 2).

1.2 Definition of additional notation in the context of word sequences. As already mentioned in 1.1, integer indexes are used to denote the ith word of a word sequence; in order to denote single letters or subwords of a word, substring operators are used as defined below. Observe that when considering words and word sequences we have to make a distinction between the enumeration of the letters on the one hand and the enumeration of the places between the letters on the other hand. The following formula shows both numbers systems when considering the Latin alphabet as a word:

$$\begin{array}{ccccccccccccccccccccccc}
 & a & & b & & c & & d & & e & & d & & e & & \cdots & & x & & y & & z & . \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & | & 1 & | & 2 & | & 3 & | & 4 & | & 5 & | & 6 & | & 7 & & 23 & | & 24 & | & 25 & | & 26 \\
 & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & & & 24 & & 25 & & 26 &
\end{array}$$

Denote by ℓ the length of a word, i.e., the number of letters in the word. Assume that ω_i, ω_{i+1} are words belonging to a word sequence ω and recall that ω_i can be obtained from ω_{i+1} by eliminating the letters α_{i+1}. Hence as the inverse of this elimination process each letter of ω_i can be associated with a letter of ω_{i+1}. By $\nu_{\omega, i}$ we denote the strictly monotonic function $\{1, 2, 3, \dots, \ell(\omega_i)\} \to \{1, 2, 3, \dots, \ell(\omega_{i+1})\}$ which maps the position number of the jth letter of ω_i to the position number of the corresponding letter of ω_{i+1} which the jth letter is associated with. Either of the two or both indices of a ν-function can be omitted if they are obvious by the context. If they are omitted, a product $\nu_{\omega, i} \circ \nu_{\omega, i-1} \circ \nu_{\omega, i-2} \circ \cdots \circ \nu_{\omega, i-k}$ is just denoted as ν^{k+1}. For technical reasons these ν-functions are extended to $\{0, 1, 2, \dots, \ell(\omega_i) + 1\} \to \{0, 1, 2, \dots, \ell(\omega_{i+1}) + 1\}$ by $\nu(0) = 0$, $\nu(\ell(\omega_i) + 1) = \ell(\omega_{i+1}) + 1$.

1.3 DEFINITION. A *position sequence* $(k_i)_{i \in \mathbb{N}}$ for a word sequence ω is an integer sequence such that $0 \leq k_i \leq \ell(\omega_i)$ and

$$\nu_i(k_i) \leq k_{i+1} \leq \nu_i(k_i + 1) - 1. \qquad (1.3.1)$$

The ith entry of such a sequence can be interpreted as the number of a place between two letters of word ω_i and hence the whole position sequence is a description of what can be seen as a place within a word sequence. Thus position sequences can be used for the definition of "sub-word-sequences"; see 1.5.

1.4. DEFINITION OF THE ORDERING OF POSITION SEQUENCES. If two position sequences $(k_i)_{i \in \mathbb{N}}$ and $(m_i)_{i \in \mathbb{N}}$ satisfy $k_i \leq m_i$ for all $i \in \mathbb{N}$, we will say $(k_i)_{i \in \mathbb{N}} \leq (m_i)_{i \in \mathbb{N}}$. This defines an ordering on the set $\mathbb{P}(\omega)$ by which we will denote the set of position sequences for the fixed word sequence ω. In contrast to the position numbers of a finite word this ordering does not permit the definition of the predecessor and successor. Nevertheless from 1.3(1) it is easy to verify that this definition gives a total ordering which has a minimal and a maximal element. The minimal element is the sequence $(0, 0, 0, \dots)$ and the maximal element is the sequence $(\ell(\omega_1), \ell(\omega_2), \ell(\omega_3), \dots)$. In the future these two position sequences will be abbreviated by $0^{\mathbb{N}}$ and $\ell^{\mathbb{N}}$.

1.5 DEFINITION OF SUBWORD SEQUENCES. (i) Let $w = \alpha_{i_1}^{\varepsilon_1} \alpha_{i_2}^{\varepsilon_2} \cdots \alpha_{i_n}^{\varepsilon_n}$ be a word and let $0 \leq k \leq l \leq n$. Then we will write $w_{|[k,l]}$ to denote the subword $\alpha_{i_{k+1}}^{\varepsilon_{k+1}} \alpha_{i_{k+2}}^{\varepsilon_{k+2}} \cdots \alpha_{i_l}^{\varepsilon_l}$ for $k < n$ and the empty word for $k = l$. Observe that this convention is compatible with understanding k and l as place numbers in the sense of 1.2.

(ii) Let ω be a word sequence and $\kappa = (k_i)_{i \in \mathbb{N}}$ and let $\mu = (m_i)_{i \in \mathbb{N}}$ be elements of $\mathbb{P}(\omega)$ with $\kappa < \mu$. Then we let $\omega_{|[\kappa, \mu]}$ be the sequence of subwords which by using the notation of (i) can be defined as follows:

$$\omega_{|[\kappa, \mu]} := \left(\omega_{i|[k_i, m_i]} \right)_{i \in \mathbb{N}}. \qquad (1.5.1)$$

By 1.3(1) it follows that $\omega_{|[\kappa, \mu]}$ again satisfies the properties of a word sequence as listed in 1.1; i.e., subword sequences are word sequences according to our definition. Conversely position sequences may be understood as the combinatorial possibilities for breaking up a word sequence in such a way that the pieces are word sequences again.

1.6. LEMMA–DEFINITION OF THE "GREATEST COMMON INITIAL WORD SEQUENCE." *Let ω, η be any two word sequences. Then there exists one greatest position sequence $\kappa \in \mathbb{P}(\omega) \cap \mathbb{P}(\eta)$ such that $\omega_{|[0^{\mathbb{N}}, \kappa]} = \eta_{|[0^{\mathbb{N}}, \kappa]}$.*

Proof. Essentially the proof will be based on looking at each index level separately, and on using the fact that for each of those levels the structure is finite and hence the existence of maxima is trivial. If such a construction is performed with suitable criteria, it gives on the one hand a valid position sequence and satisfies on the other hand the required maximality criteria. In that spirit the quickest way of obtaining the greatest common initial word sequence is considering the set of position sequences

$$\Lambda = \left\{ \lambda \in \mathbb{P}(\omega) \cap \mathbb{P}(\eta) \mid \omega_{|[0^{\mathbb{N}}, \lambda]} = \eta_{|[0^{\mathbb{N}}, \lambda]} \right\},$$

which is not empty since it contains $0^{\mathbb{N}}$. Since Λ is only composed of valid position sequences, $k_i := \max\{\lambda_i \mid \lambda \in \Lambda\}$ also satisfy 1.3(1) and hence give a position sequence $\kappa := (k_i)_{i \in \mathbb{N}}$, and it obviously satisfies the desired maximality property. ∎

An alternate but more lengthy proof that perhaps gives a better understanding of the problems behind the existence of greatest common initial word sequences may be obtained by analyzing the procedure described below for finding such sequences:

1.7 PROCEDURE FOR FINDING THE GREATEST COMMON INITIAL WORD SEQUENCE κ OF TWO WORD SEQUENCES ω AND η. The process for finding κ consists of repeating the steps (i) and (ii) $\#(\mathbb{N})$ times such that at the mth repetition level a subword of ω_m is compared with a subword of η_m, and, depending on the results of these comparisons, k_m is defined. When initializing this process the complete words ω_1 and η_1 are to be considered as the words to be compared:

(i) Let $\omega_{m|[a,b]}$ and $\eta_{m|[a,b']}$ be the words to be compared at the mth level. Then for all integers $i \in \{a+1, a+2, \dots, \min\{b, b'\}\}$ we ask the following questions in the natural order given by the i-values:

$$\text{Is } \omega_{m+j|[\nu^j(i-1), \nu^j(i)]} = \eta_{m+j|[\nu^j(i-1), \nu^j(i)]} \qquad \text{for all } j \in \mathbb{N} \cup \{0\}?$$

If the answer to all the $\min\{b, b'\} - a$ questions is "yes," we let $k_m := \min\{b, b'\}$. Otherwise k_m is defined as the predecessor of the number of the i-value of the first question that was answered by "no."

(ii) When passing from the mth to the $(m+1)$st level we specify $\omega_{m+1|[\nu(k_m), \nu(k_m+1)-1]}$ and $\eta_{m+1|[\nu(k_m), \nu(k_m+1)-1]}$ to be the words to be compared for the $(m+1)$st level. Note that in this context we have to distinguish between the ν-functions ν_ω and ν_η (cf. 1.2), and that the construction process guarantees that $\nu_\omega(k_m) = \nu_\eta(k_m)$, but not that $\nu_\omega(k_m + 1) = \nu_\eta(k_m + 1)$.

1.8 DEFINITION. The *juxtaposition of two word sequences* ω and η is $(\omega_i \eta_i)_{i \in \mathbb{N}}$, i.e, is defined as the word sequence in which the ith word is obtained by attaching the corresponding words of ω and η to each other without any cancellations. Further we define ω^{-1}, the *inverse of the word sequence* ω, as $(\omega_i^{-1})_{i \in \mathbb{N}}$, i.e., as the word sequence in which the ith word is obtained from ω_i by writing the letters in the inverse order and negating the exponent of each letter. No cancellations are to be performed, either. Besides this, we establish the following notation for the *inverse of a position sequence*: If $\kappa \in \mathbb{P}(\omega)$, we define $\kappa^{-1} \in \mathbb{P}(\omega^{-1})$ such that the places in ω and ω^{-1} to which κ and κ^{-1} point, respectively, correspond to each other with respect to the inversion process, i.e., $\kappa^{-1} = (\ell(\omega_i) - k_i)_{i \in \mathbb{N}}$.

1.9 DEFINITION. A word sequence ω is called *reduced* or *short* if, whenever a two-lettered subword of some ω_i consists of two inverse symbols, i.e., whenever we have $\omega_{i|[k-1,k+1]} = \alpha_i^{\varepsilon} \cdot \alpha_i^{-\varepsilon}$, there exists $j \in \mathbb{N}$ such that $\omega_{i+j|[\nu^j(k)-1,\nu^j(k+1)]}$ is not equivalent to the empty word by elementary cancellations.

1.10 EXAMPLE FOR A WORD SEQUENCE. Let

$$\begin{aligned}
\omega_1 &= \alpha_1 \alpha_1^{-1} \\
\omega_2 &= \alpha_1 \alpha_2 \alpha_1^{-1} \alpha_2^{-1} \alpha_2 \\
\omega_3 &= \omega_2 \\
\omega_4 &= \alpha_1 \alpha_2 \alpha_1^{-1} \alpha_2^{-1} \alpha_4 \alpha_2 \alpha_4^{-1} \alpha_4 \\
\omega_5 &= \omega_6 = \omega_7 := \omega_4 \\
\omega_8 &= \alpha_1 \alpha_2 \alpha_1^{-1} \alpha_2^{-1} \alpha_4 \alpha_2 \alpha_4^{-1} \alpha_8 \alpha_4 \alpha_8^{-1} \alpha_8 .
\end{aligned} \tag{1.10.1}$$

This construction principle is iterated. For all i which are not powers of two ω_i just follows from ω_{i-1} by not performing any insertions. If $i = 2^j$, we have that ω_i results from inserting one letter α_i into ω_{i-1} at the penultimate position, and from then appending $\alpha_i^{-1}\alpha_i$ to the end of that word. Since the conditions of 1.1 are obviously satisfied, ω as defined in (1) is a reduced word sequence, although none of the words actually is reduced. Observe that the classical way of representing this element as an $\varprojlim(F_n)$-element would be to use reduced words only; i.e., according to that style the following sequence would have been used:

$$\begin{aligned} g_1 &= 1 \in \langle \alpha_1 \mid - \rangle \\ g_2 &= \alpha_1 \alpha_2 \alpha_1^{-1} \in \langle \alpha_1, \alpha_2 \mid - \rangle \\ g_3 &= \alpha_1 \alpha_2 \alpha_1^{-1} \in \langle \alpha_1, \alpha_2, \alpha_3 \mid - \rangle \\ g_4 &= \alpha_1 \alpha_2 \alpha_1^{-1} \alpha_2^{-1} \alpha_4 \alpha_2 \\ g_5 &= g_6 = g_7 := g_4 \\ g_8 &= \alpha_1 \alpha_2 \alpha_1^{-1} \alpha_2^{-1} \alpha_4 \alpha_2 \alpha_4^{-1} \alpha_8 \alpha_4 . \end{aligned} \tag{1.10.2}$$

This example in particular illustrates the phenomenon that some letters α_i, that by the axioms of a word sequence have to appear for the first time in the $(\omega_j)_{j \in \mathbb{N}}$-sequence on the ith level, may first show up in the $(g_j)_{j \in \mathbb{N}}$-sequence for a considerably higher index level. The above example is constructed in such a way that this difference of the index levels of the first appearances of α_i is i; i.e., it is given by an unbounded sequence for the overall word sequence. Observe that this delay of the appearances of letters can mean that approximation constructions such as 1.14 can only be done in a less straightforward way if they have to be performed with respect to reduced $\varprojlim (F_n)$-sequences instead of word sequences.

1.11 DEFINITION OF THE PRODUCT OF REDUCED WORD SEQUENCES. Let G be the set of all reduced word sequences on the alphabet $\mathbb{A} = \{\alpha_1, \alpha_2, \ldots\}$. Then the multiplication "$\cdot$" is defined as the composition $G \times G \to G$,

$$\omega \cdot \eta := \omega_{|[0^{\mathbb{N}}, \kappa^{-1}]} \eta_{|[\kappa, \ell^{\mathbb{N}}]}, \tag{1.11.1}$$

where κ is defined such that $\eta_{|[0^{\mathbb{N}}, \kappa]}$ and $\omega^{-1}_{|[0^{\mathbb{N}}, \kappa]}$ are the greatest common initial word sequences of η and ω^{-1} in the sense of 1.6. Observe that the composition on the right hand side of the formula displayed above was just the juxtaposition defined in 1.8. For a proof that the right hand side of (1) actually is a reduced word sequence, see (ii) in the final paragraph of [Zas, Sec. 2]. We note the following properties which are immediate by definition:

(i) There is only one word sequence that acts as neutral element, namely, the empty one.

(ii) A product between a word sequence and its inverse as defined in 1.8 gives the empty word sequence.

As already indicated in the Introduction, the group of all word sequences can be geometrically interpreted as the fundamental group of the Hawaiian Earrings.

1.12 THEOREM (see [Zas, 2.1–2.3]). *G, provided with the composition of* 1.11, *is a group which is isomorphic to the fundamental group of the Hawaiian Earrings. The natural isomorphism is induced by the mapping $G \to C^0(I, Y)$, $\omega \mapsto f_\omega$, where the curve f_ω: $I \to Y$ is defined by Process* 1.14; *see below.*

The proof of Theorem 1.12 will occupy most paragraphs of the remainder of this section and will not be complete before 1.20.

1.13 *Remark.* By definition, elements of G are to be described just by reduced word sequences; i.e., within the set of G-elements there is no equivalence relation defined by the classical cancellation processes. Consequently, the product rule 1.11(1) gives the only framework in which within G-computations cancellation between word sequences takes place.

1.14 *Process listing.* Let there be given a word sequence ω; a curve f_ω: $I \to Y$ will be defined by iterating the steps below infinitely many times, such that at each interation step a subinterval of the parameter domain of f_ω is considered and a definition of f_ω is obtained on some subintervals of that subinterval. During this process a subword of the word sequence ω is associated to each interval of consideration. When initializing this process, the interval to be considered is I (the complete parameter domain of f_ω) and it is associated with the word ω_1.

(1) Let $\omega_{m|[a,b]}$ be the subword of consideration. Establish a uniform subdivision of the interval of consideration in

$$(b - a) + \#\Big\{i \in \{a, a+1, a+2, \dots, b\} \mid \omega_{m+j|[\nu^j(i),\, \nu^j(i+1)-1]} \text{ is not empty for all } j \in \mathbb{N}\Big\}$$

intervals, so that each of these intervals is associated either with a letter of ω_m or with a place between the letters of ω_m where there are insertions of other letters for higher indexed ω_{m+j}.

(2) For each of those intervals I_n resulting from the subdivision in Step (1) that are associated with a letter of ω_m we define the restriction $f_{\omega|I_n}$ to that interval: $f_{\omega|I_n}$ results from a linear parametrization as a mapping that maps $I_n/\partial I_n$ to the mth loop of the Hawaiian Earrings as a degree 1- or (-1)-mapping depending on whether the corresponding letter of ω_m is α_m or α_m^{-1}, respectively. The endpoints of I_n are mapped to the accumulation point of the Hawaiian Earrings.

(3) The infinitely many repetitions of Steps (1) and (2) of this process consist of a finite number of repetitions of it on each "level" before passing to the next level. The levels are denumerated by $\mathbb{N}$, and on the mth level restrictions of f_ω with values lying in the mth loop of the

Hawaiian Earrings are defined from considering subwords of ω_m. The finitely many repetitions of the process on one level result from considering all intervals of I one after the other on which f_ω has not been defined in the preceding levels. During the mth level each of these intervals is subdivided and definitions of f_ω are established on some of the new intervals. The remaining intervals are associated with a place between the letters of ω_m. On the $(m+1)$st level these remaining intervals become the intervals of consideration and, if c denotes the number of the place of association between the letters of ω_m for some interval, this interval is associated with $\omega_{m+1|[\nu(c),\,\nu(c+1)-1]}$ on the $(m+1)$st level.

1.15 *The scheme of proof for Theorem* 1.12. Even though word sequences are a suitable concept for proving Theorem 1.12 in a self-contained way, for the sake of brevity we will use a way that needs the references to Griffiths and Morgan and Morrison. They have shown in [Gr2, Theorem 6.3; MM, Theorem 4.1] that $\pi_1(Y)$ can be described by a certain subgroup of $\varprojlim(F_n)$, and in this paper we will only prove that the group of all reduced word sequences can be naturally embedded into $\varprojlim(F_n)$ having precisely the same image. Observe that the condition "locally eventually constant" of [MM] (that implicitly is also used in [Gr1, Gr2]) is the same as our condition $b_i(g) < \infty$ for all $i \in \mathbb{N}$ (cf. the Introduction). The procedure as described in 1.14 results from combining our embedding with the isomorphisms as constructed by [Gr1, Gr2, MM].

Our embedding of the group of reduced word sequences into $\varprojlim(F_n)$ is natural since the ith word of a word sequence can be interpreted as an element of F_i. Further observe that the compatibility condition of 1.1 that makes a sequence of words a "word sequence" (i.e., trivializing α_n) precisely corresponds to the condition that makes a sequence of group elements an $\varprojlim(F_n)$-sequence. Also note that our product-definition 1.11(1) is compatible with the product that is defined for an inverse limit. Hence only two facts need to be proved to order to obtain our Theorem 1.12, namely

(i) that the combinatorially defined product for reduced word sequences is associative (and hence that the reduced word sequences actually form a group, cf. 1.11(i) and (ii), and

(ii) that the above described natural homomorphism to map reduced word sequences to $\varprojlim(F_n)$-sequences has a trivial kernel and hence is actually an embedding.

The proofs of (i) and (ii) will be given in 1.19 and 1.20, respectively. However, although it will not actually be clear before 1.20 that the reduced word sequences actually form a group, we will call them "group" from now on, since this allows us to state Lemma 1.16 in a way that is less irritating when it is used as a reference in the remainder of this paper. In particular note that these lemmata do also make sense if the product is regarded as possibly non-associative and hence must be given with a concrete parenthesization.

Roughly speaking, the following lemma states that all phenomena of performing the multiplication of reduced word sequences (i.e., of infinite objects) reduce to phenomena of finitely generated free groups, provided one establishes a corresponding subdivision for each of the word sequences. More precisely we will prove:

1.16 LEMMA. *Let* $\omega_1, \dots, \omega_k$ *denote reduced word sequences. Then, if* $[0^{\mathbb{N}}, \ell_j^{\mathbb{N}}]$ *denotes the interval of position sequences of* ω_j, *there exists a finite subdivision* $0^{\mathbb{N}} = a_{j,0} < a_{j,1} < \cdots < a_{j,j_k} = \ell_j^{\mathbb{N}}$ *of each of these intervals such that, if the subsequently described Procedure* 1.17 *is applied to this subdivision, the result is the reduced word sequence that according to* 1.11(1) *gives the product* $\omega_1 \cdot \cdots \cdot \omega_k$.

1.17 *Procedure listing.* Let there be given a finite number of reduced word sequences and for each (of those word sequences let there be given) a possibly empty set of position sequences that will be denoted as in Lemma 1.16. Denote each of the word sequences $\omega_{i|[a_{i,j-1}, a_{i,j}]}$ by a symbol using the same symbols for identical word sequences and inverse symbols for inverse ones. Then replace the product $\omega_1 \cdot \cdots \cdot \omega_k$ by the corresponding product of those symbols. Now start to reduce the latter product according to the calculus used for free generators of groups. Replace the symbols that still occur in the reduced product by the corresponding word sequences and concatenate these word sequences. The result is the output of this procedure.

1.18 *Proof of Lemma* 1.16. The Lemma now follows by induction on the length k of the given product.

Zero step. The cases $k = 0$ and $k = 1$ are trivial. $k = 2$ then follows immediately from Formula 1.11(1), since according to 1.11(1) the product of two reduced word sequences ω and η is to be computed by concatenating $\omega_{|[0^{\mathbb{N}}, a]}$ and $\eta_{|[b, \ell^{\mathbb{N}}]}$, where a and b have been defined so that the greatest common initial word sequence of η and ω^{-1} is assumed to be $\eta_{|[0^{\mathbb{N}}, b]} = (\omega_{|[a, \ell^{\mathbb{N}}]})^{-1}$. This fits into the framework of Procedure 1.17.

Inductive step. Let there be given a product of reduced word sequences

$$(\omega_1 \cdot \omega_2 \cdot \cdots \cdot \omega_m) \cdot (\omega_{m+1} \cdot \cdots \cdot \omega_k) \tag{1.18.1}$$

and assume that the product is parenthesized so that the outermost pairs of parentheses are placed as displayed in Formula (1). In the following we want to

(I) introduce further relevant notation in order to be able to

(II) precisely describe what the reduced word sequence that results from performing the multiplication of (1) looks like. This will enable us to

(III) conclude that the word sequence (1) can also be reached by a process as described in 1.17.

Ad (I). To have a basic reference for all positions that need to be referred to in this inductive step (cf. the first paragraph of the proof of 1.19), we hereby define

$$\omega := \omega_1 \omega_2 \cdots \omega_m \omega_{m+1} \cdots \omega_k; \tag{1.18.2}$$

i.e., ω is the mere concatenation of all factors, and for the remainder of 1.18 positions are to be described by $\mathbb{P}(\omega)$-sequences, unless something else is explicitly stated. We obtain

$$\omega_1 \omega_2 \cdots \omega_m = \omega_{|[0^{\mathbb{N}}, \kappa']} \qquad \text{and} \qquad \omega_{m+1} \omega_{m+2} \cdots \omega_k = \omega_{|[\kappa', \ell^{\mathbb{N}}]}$$

by defining $\kappa' \in \mathbb{P}(\omega)$ appropriately. By applying the induction hypothesis to each of these factors, it follows that each of the pairs of parentheses of (1) can be described as an appropriate concatenation of subword sequences of ω. By using the undashed symbol κ and appropriate indices for denoting the corresponding cut-positions by elements of $\mathbb{P}(\omega)$, 1.17 gives us a partition

$$\begin{aligned} 0^{\mathbb{N}} \le \kappa_1 < \kappa_2 < \kappa_3 < \cdots < \kappa_{2\cdot i} \le \kappa' \le \kappa_{2\cdot i+1} < \kappa_{2\cdot i+2} < \cdots \\ < \kappa_{2\cdot j} \le \ell^{\mathbb{N}} \end{aligned} \tag{1.18.3}$$

such that

$$\omega_1 \cdot \omega_2 \cdot \cdots \cdot \omega_m = \omega_{|[\kappa_1, \kappa_2]} \omega_{|[\kappa_3, \kappa_4]} \cdots \omega_{|[\kappa_{2\cdot i-1}, \kappa_{2\cdot i}]} =: \omega' \tag{1.18.4}$$

and such that

$$\omega_{m+1} \cdot \omega_{m+2} \cdot \cdots \cdot \omega_k = \omega_{|[\kappa_{2\cdot i+1}, \kappa_{2\cdot i+2}]} \omega_{|[\kappa_{2\cdot i+3}, \kappa_{2\cdot i+4}]} \cdots \omega_{|[\kappa_{2\cdot j-1}, \kappa_{2\cdot j}]} =: \omega''. \tag{1.18.5}$$

Ad (II). By 1.11(1) and the parenthesizing of (1), the product (1) has to be evaluated as

$$\omega'_{|[0^{\mathbb{N}}, \kappa^{-1}]} \cdot \omega''_{|[\kappa, \ell^{\mathbb{N}}]} \tag{1.18.6}$$

with $\kappa \in \mathbb{P}(\omega'') \cap \mathbb{P}(\omega'^{-1})$. By returning to the convention as set in (I) of denoting all positions by $\mathbb{P}(\omega)$-sequences, we let κ'', κ''' be the $\mathbb{P}(\omega)$-sequences that point at the same positions as are described by κ^{-1}, κ as $\mathbb{P}(\omega')$- or $\mathbb{P}(\omega'')$-sequences, respectively. In particular we can choose

$$\begin{aligned} &\kappa'' \in [\kappa_1, \kappa_2) \cup [\kappa_3, \kappa_4) \cup \cdots \cup [\kappa_{2\cdot i-3}, \kappa_{2\cdot i-2}) \cup [\kappa_{2\cdot i-1}, \kappa_{2\cdot i}] \\ \text{and} \quad &\kappa''' \in [\kappa_{2\cdot i+1}, \kappa_{2\cdot i+2}] \cup (\kappa_{2\cdot i+3}, \kappa_{2\cdot i+4}] \cup \cdots \cup (\kappa_{2\cdot j-1}, \kappa_{2\cdot j}]. \end{aligned} \tag{1.18.7}$$

The above formulae just reflect the fact that the positions κ' and κ'' can only lie in those parts of ω that have not been cut away in the course of computing the products (4) and (5), while by using semi-open intervals a convention is given in which $\mathbb{P}(\omega)$-sequences are to be used for describing those positions inside ω' or ω'' that result from having cut away some part of ω. If $\kappa'' = \kappa_{2\cdot i}$ then $i' := i$; otherwise we choose i' such that $\kappa'' \in [\kappa_{2\cdot i'-1}, \kappa_{2\cdot i'})$. Similarly we either have that $\kappa''' \in (\kappa_{2\cdot j'-1}, \kappa_{2\cdot j'}]$ or that $\kappa''' = \kappa_{2\cdot j'-1}$ and $j' = i + 1$. By having introduced this notation we finally get that the product (1) can be described by concatenations of parts of ω as follows:

$$\omega_{|[\kappa_1, \kappa_2]}\omega_{|[\kappa_3, \kappa_4]} \cdots \omega_{|[\kappa_{2\cdot i'-1}, \kappa'']}\omega_{|[\kappa''', \kappa_{2\cdot j'}]}\omega_{|[\kappa_{2\cdot j'+1}, \kappa_{2\cdot j'+2}]} \cdots \omega_{|[2\cdot j-1, 2\cdot j]}. \tag{1.18.8}$$

The parts of ω that have been cut away in Step (8) of computing (1), i.e.,

$$\begin{aligned} \omega'_{|[\kappa^{-1}, \ell^{\mathbb{N}}]} &:= \omega_{|[\kappa'', \kappa_{2\cdot i'}]}\omega_{|[\kappa_{2\cdot i'+1}, \kappa_{2\cdot i'+2}]} \cdots \omega_{|[2\cdot i-1, 2\cdot i]} \qquad \text{and} \\ \omega''_{|[0^{\mathbb{N}}, \kappa]} &:= \omega_{|[\kappa_{2\cdot i+1}, \kappa_{2\cdot i+2}]}\omega_{|[\kappa_{2\cdot i+3}, \kappa_{2\cdot i+4}]} \cdots \omega_{|[\kappa_{2\cdot j'-1}, \kappa''']}, \end{aligned}$$

are inverse to each other. Hence each of the attaching positions of one of these subword sequences has a “mirror-image” (cf. the proof of 1.19) in the other word sequence. Although we will not introduce new symbols in this text for describing these mirror-image positions, we should think of having defined some kind of variables for reference in the forthcoming computations. The non-uniqueness that we had experienced when introducing κ'' and κ''' and that we had overcome in (7) by working with semi-open intervals is not a problem in this case since, if the mirror-image of an attaching point should be another attaching point, there is no need to newly associate a variable with this position.

Ad (III). The desired process whose existence was claimed in 1.16 can now be described as follows: The set of all positions that is to be used results from the union of $\{0^{\mathbb{N}}, \kappa_1, \ldots, \kappa_{2\cdot j}, \ell^{\mathbb{N}}\}$ (see (3)) with the mirror-images as described in the final paragraph of (II). First we perform all those cancellations that belong to computing the subproducts (4) and (5). Since these cancellations only take place in the first or in the second half of our word sequence, respectively, they do not interfere with each other. In addition note that these processes are not affected by having introduced our mirror-images as additional cut-points, since mirror-images are only defined in those parts of the word sequence ω that survive in the above cancellation process. Performing these cancellations leaves us with the word sequence $\omega'\omega''$ and with the task of still cancelling it down to the form of (6). However, this can now be performed: By having introduced our mirror-images as additional cut-points, $\omega'_{|[\kappa^{-1}, \ell^{\mathbb{N}}]}$ and $\omega''_{|[0^{\mathbb{N}}, \kappa]}$ are subdivided into matching parts of word sequences which are mutually inverse to each other. Hence these parts also cancel against each other in the framework given in 1.17, leaving us with the desired result. ∎

1.19 Lemma. *The product of reduced word sequences as defined in* 1.11(1) *is associative*.

Proof. Let ω be the (not necessarily reduced) word sequence that results from concatenating three different word sequences. Let $\kappa_3 < \kappa_6 \in \mathbb{P}(\omega)$ denote the attaching places of the first two and the last two of our three factors, respectively. That way $\omega_{|[0^{\mathbb{N}}, \kappa_3]}$, $\omega_{|[\kappa_3, \kappa_6]}$, and $\omega_{|[\kappa_6, \ell^{\mathbb{N}}]}$ describe the three factors of our concatenation; because our concept of position sequences (cf. 1.3) is based on counting the number of letters of the beginning and hence is not compatible with the concatenation or multiplication of word sequences, we will not introduce separate variables for denoting these factors. Instead of this we will in the following proof describe all word sequences as substrings of ω and thus use ω as the only reference of identifying positions; i.e., all subsequently used position sequences are to be understood as $\mathbb{P}(\omega)$-elements.

Let $\kappa_5 \in [\kappa_3, \kappa_6]$ bound the greatest common initial word sequence of $\omega_{|[\kappa_3, \kappa_6]}$ and $(\omega_{|[0^{\mathbb{N}}, \kappa_3]})^{-1}$. Denote by $\kappa_1 \in [0^{\mathbb{N}}, \kappa_3]$ the corresponding *mirror-image* of κ_5; i.e., we have that $\kappa_3 - \kappa_1 = \kappa_5 - \kappa_3$, where these differences of position sequences are defined as those sequences of integers which result from computing the differences termwise. Hence in the sense of 1.11(1) we have that $\omega_{|[0^{\mathbb{N}}, \kappa_3]} \cdot \omega_{|[\kappa_3, \kappa_6]} = \omega_{|[0^{\mathbb{N}}, \kappa_1]}\omega_{|[\kappa_5, \kappa_6]}$. Similarly we define $\kappa_8 \in [\kappa_6, \ell^{\mathbb{N}}]$ as the bound of the greatest common initial word sequence of $\omega_{[\kappa_6, \ell^{\mathbb{N}}]}$ and of $(\omega_{|[\kappa_3, \kappa_6]})^{-1}$, and κ_4 as its mirror image within $[\kappa_3, \kappa_6]$ so that we obtain that $\omega_{|[\kappa_3, \kappa_6]} \cdot \omega_{[\kappa_6, \ell^{\mathbb{N}}]} = \omega_{|[\kappa_3, \kappa_4]}\omega_{[\kappa_8, \ell^{\mathbb{N}}]}$. The case where $\kappa_4 > \kappa_5$ is the easy case of this proof, since then $\omega_{|[0^{\mathbb{N}}, \kappa_3]} \cdot \omega_{|[\kappa_3, \kappa_6]} \cdot \omega_{[\kappa_6, \ell^{\mathbb{N}}]} = \omega_{|[0^{\mathbb{N}}, \kappa_1]}\omega_{|[\kappa_5, \kappa_4]}\omega_{|[\kappa_8, \ell^{\mathbb{N}}]}$ independently of how the product is parenthesized. Hence for the remainder of the proof we will assume that $\kappa_4 \leq \kappa_5$. In this case we define two new position sequences:

$\kappa_2 \in [\kappa_1, \kappa_3]$ results from using $(\omega_{|[\kappa_1, \kappa_3]})^{-1} = \omega_{|[\kappa_3, \kappa_5]}$ and by mirroring κ_4 into $[\kappa_1, \kappa_3]$, and similarly from the fact that $(\omega_{|[\kappa_4, \kappa_6]})^{-1} = \omega_{|[\kappa_6, \kappa_8]}$ gives that κ_5 has a mirror-image within $[\kappa_6, \kappa_8]$ which shall be denoted by κ_7. From the above relations we obtain that on the level of not necessarily reduced word sequences we have that $\omega_{|[0^{\mathbb{N}}, \kappa_2]}\omega_{|[\kappa_8, \ell^{\mathbb{N}}]} = \omega_{|[0^{\mathbb{N}}, \kappa_1]}\omega_{|[\kappa_7, \ell^{\mathbb{N}}]}$, since

$$\omega_{|[\kappa_1, \kappa_2]} = (\omega_{|[\kappa_4, \kappa_5]})^{-1} = \omega_{|[\kappa_7, \kappa_8]}. \tag{1.19.1}$$

The right hand side of this equation results (at least, as a possibly intermediate step) from computing $(\omega_{|[0^{\mathbb{N}}, \kappa_3]} \cdot \omega_{|[\kappa_3, \kappa_6]}) \cdot \omega_{|[\kappa_6, \ell^{\mathbb{N}}]}$, namely, by completely computing the first product and by at least performing those cancellations of the second multiplication that according to our assumed symmetries can be performed. Similarly the left hand side results from a possibly intermediate step of computing $\omega_{|[0^{\mathbb{N}}, \kappa_3]} \cdot (\omega_{|[\kappa_3, \kappa_6]} \cdot \omega_{|[\kappa_6, \ell^{\mathbb{N}}]})$. However, apart from a limit case that is discussed below, we also see that these coinciding possibly intermediate steps are actually reduced and hence give matching results of computing $(\omega_{|[0^{\mathbb{N}}, \kappa_3]} \cdot \omega_{|[\kappa_3, \kappa_6]}) \cdot \omega_{|[\kappa_6, \ell^{\mathbb{N}}]}$, $\omega_{|[0^{\mathbb{N}}, \kappa_3]} \cdot (\omega_{|[\kappa_3, \kappa_6]} \cdot \omega_{|[\kappa_6, \ell^{\mathbb{N}}]})$, respectively. This follows, since the attaching point of $\omega_{|[0^{\mathbb{N}}, \kappa_2]}$ and $\omega_{|[\kappa_8, \ell^{\mathbb{N}}]}$ looks precisely like the inner position of κ_8 of the reduced factor $\omega_{|[\kappa_6, \ell^{\mathbb{N}}]}$ and hence does not permit further reductions, and similarly for the attaching point of the other concatenation that is the inner point κ_1 of the reduced factor $\omega_{|[0^{\mathbb{N}}, \kappa_3]}$. This gives our proof of associativity—apart from the above mentioned limit case. This case is given when

$$\kappa_1 = \kappa_2 \qquad \text{and} \qquad \kappa_4 = \kappa_5 \qquad \text{and} \qquad \kappa_7 = \kappa_8$$

(cf. (1)), since in that case we do not find κ_1 and κ_8 as inner positions of reduced word sequences. Hence in that case our result $\omega_{|[0^{\mathbb{N}}, \kappa_1]}\omega_{|[\kappa_7, \ell^{\mathbb{N}}]} = \omega_{|[0^{\mathbb{N}}, \kappa_2]}\omega_{|[\kappa_8, \ell^{\mathbb{N}}]} = \omega_{|[0^{\mathbb{N}}, \kappa_1]}\omega_{|[\kappa_8, \ell^{\mathbb{N}}]}$ might not be reduced and might actually just be an intermediate step in the computation of our two three-factor products. However, according to the way these two products are to be computed, the coincidence of these intermediate steps implies that the end results will also coincide: If $\omega_{|[0^{\mathbb{N}}, \kappa_1]}\omega_{|[\kappa_8, \ell^{\mathbb{N}}]}$ should not be reduced, the greatest common initial word sequence of $\omega_{|[\kappa_8, \ell^{\mathbb{N}}]}$ and of $(\omega_{|[0^{\mathbb{N}}, \kappa_1]})^{-1}$ is not empty and is bounded by two position sequences $\kappa_9 \in (\kappa_8, \ell^{\mathbb{N}}]$ and $\kappa_0 \in [0^{\mathbb{N}}, \kappa_1)$. For both of our three-factor products the second computation step in the sense of 1.11(1) will find these position sequences κ_0 and κ_9 as bounds of the coinciding part and hence will come up with the result $\omega_{|[0^{\mathbb{N}}, \kappa_0]}\omega_{|[\kappa_9, \ell^{\mathbb{N}}]}$ in both cases. ∎

1.20 LEMMA. *The natural homomorphism as described in* 1.15 *for mapping the group of reduced word sequences into* $\varprojlim(F_n)$ *has a trivial kernel.*

Proof. An element of the kernel would be a word sequence $\omega = (\omega_j)_{j \in \mathbb{N}}$ where each ω_i can be cancelled by elementary reductions through to the empty word and which is yet reduced in the sense of 1.9. By assuming that our kernel is not trivial, we assume that ω is not the empty word sequence. Let i be the smallest index such that ω_i is not empty. By combining what follows from the insertion structure of a word sequence (cf. 1.1) with the above assumption of being completely cancellable, it follows that ω_i must consist of the same number of letters α_i and α_i^{-1}. We now in particular consider an innermost cancelling pair $\alpha_i^{\pm 1}\alpha_i^{\mp 1}$. The lemma will essentially follow by arguing that all the higher indexed letters that might be inserted at this place must form a completely cancellable subword $\omega'_{i+j} = \alpha_i^{\pm 1} \cdots \alpha_i^{\mp 1}$ where the dots only replace letters α_k, α_k^{-1} with $k > i$; we want to conclude that ω'_{i+j} is completely cancellable since the entire word ω_{i+j} has by assumption this property and since the first and the last letter of ω'_{i+j} cancel against each other. What makes the above arguments not entirely complete (yet) is the possibility that our word ω_{i+j} might be cancelled in such a way to the empty word that the first letter of ω'_{i+j} is cancelled against some letter to the left, while on the other hand the last letter of ω'_{i+j} is cancelled against some letter that comes after ω'_{i+j}. Hence, in order to make the above arguments complete, one has first to find a "compatible" cancellation pattern where the phenomenon as described in the preceding sentence does not occur any more. The following two facts imply that if some cancellation pattern exists, also compatible patterns can be found:

(i) Only finitely many cancellation patterns (i.e., sequences of elementary cancellation processes) exist that can be applied to the finitely many letters α_i and α_i^{-1} that occur in ω_i (and that then are copied to all higher indexed words of ω).

(ii) Each cancellation pattern for some word ω_n naturally induces some cancellation pattern for a word ω_l with $l < n$ since ω_l is obtained from ω_n by having taken out letters in such a way that either both or none of the letters of a cancelling pair are taken out.

See the proof of [Zas, Prop. 2.10] for a graph-theoretical picture to visualize the construction of a compatible cancellation pattern out the properties (i) and (ii). Since compatible patterns exist, the lemma actually can be deduced as described above.

According to 1.15(i) and (ii) the proof of Theorem 1.12 is now complete. ∎

1.21 *Remark.* The purpose of this remark is to show how $\mathbb{P}(\omega)$ can be geometrically interpreted.

Use the bijection between G and $\pi_1(Y) = \pi_1(Y, P)$ as established in Theorem 1.12 and the fact that by 1.5(ii) $\mathbb{P}(\omega)$ is associated to the facilities of splitting up ω into two subword sequences. The analogous possibility of splitting up a $\pi_1(Y)$-element is always given when such an element is represented by a path that does not have nullhomotopic subloops (e.g., a path as it is constructed in 1.14) and this path passes through the base-point P of Y. Since "no cancellation in 1.11(1)" precisely corresponds to the situation that the concatenation of two paths that both do not have nullhomotopic subsegments results in a path that still does not have some, it is not hard to verify that the isomorphism as constructed in 1.12 is compatible with these two splitting up facilities. That way it follows that if some homotopy class ω of $\pi_1(Y)$ is represented by a path $u\colon [0,1] \to Y$ without nullhomotopic subsegments, then $\mathbb{P}(\omega)$ is bijectively associated to $\mathscr{P}(u) = u^{-1}(\{P\}) \subset [0,1]$. Hence $\mathbb{P}(\omega)$ and its order can be geometrically represented by some subset of the reals, namely by $\mathscr{P}(u)$.

1.22 Convention for our notation. We agree that the substring notation as introduced in 1.5(1) shall be used regardless of whether the position sequences $\kappa, \mu \in \mathbb{P}(\omega)$ satisfy $\kappa < \mu$ or not. If $\kappa > \mu$ then this notation shall be understood as $\omega^{-1}_{\|[\mu^{-1}, \kappa^{-1}]}$, and if $\kappa = \mu$ this notation describes the empty word sequence. Note that with this convention we have $\omega_{\|[\kappa, \mu]} = \omega_{\|[\kappa, \nu]} \cdot \omega_{\|[\nu, \mu]}$ for any $\kappa, \mu, \nu \in \mathbb{P}(\omega)$, regardless of whether ν is smaller, bigger, or equal with respect to κ or to μ.

2. THE SELECTION OF A BASIS FOR THE GROUP OF BOUNDED WORD SEQUENCES

In the remainder of this paper all word sequences are regarded as reduced, if the opposite is not explicitly stated (cf. 1.9). This convention is compatible with the definition of G and of its subgroup H in 1.11 and 1.12, 0.1, respectively. Non-reduced word sequences are only discussed as intermediate states that occur in the course of computing products.

2.1 Definition of coincidence patterns. Let ω_1, ω_2 be two word sequences and let $a < b$ and $c < d$ be position sequences of ω_1 and ω_2, respectively. Then we say that $\omega_{1\|[a,b]}$ is a *coinciding pattern* with $\omega_{2\|[c,d]}$ if either $\omega_{1\|[a,b]} = \omega_{2\|[c,d]}$ or $\omega_{1\|[a,b]} = (\omega_{2\|[c,d]})^{-1}$, and if the intervals $[a,b]$ and $[c,d]$ are maximal in the sense that there does not exist a position sequence $a' < a$ nor some $b' > b$ such that for suitable $c' < c$ and $d' > d$ we could get one of the following equations: $\omega_{1\|[a',b]} = \omega_{2\|[c',d]}$, $\omega_{1\|[a',b]} = (\omega_{2\|[c,d']})^{-1}$, $\omega_{1\|[a,b']} = \omega_{2\|[c,d']}$, or $\omega_{1\|[a,b']} = (\omega_{2\|[c',d]})^{-1}$.

Remark. Recall that in 1.6 and in [Zas, 2.9] when defining the "greatest common initial word sequences" we developed a technique of tracing coinciding domains of word sequences and showed that the boundary of such coinciding domains always exists as a well-defined position sequence. Hence, wherever one letter occurs in both word sequences ω_1 and ω_2, this occurrence is part of a coinciding pattern between ω_1 and ω_2, since the techniques of [Zas] can be used to uniquely determine the initial and the terminal position sequences a, b, c, and d that meet the maximality properties as required above for a coincidence pattern. However, the shortest well-defined coincidence pattern consists of one letter only.

2.2 Definition. In analogy to in 2.1 we define *internal coincidence patterns*; i.e., we similarly mark those patterns which occur at different places in the same word sequence.

2.3 *Construction of the "main list" for the group H of bounded word sequences*. In this paragraph we introduce a concept which may be understood as the main technical preparation for selecting candidates for free bases of groups of word sequences in 2.7. In general this concept will succeed in giving free generating systems of word sequences, if it is applied to a group of bounded word sequences.

The *main list* $\mathscr{M}$ is defined as a set of quintuples

$$(\iota(\omega), \omega, C(\omega), E(\omega), F(\omega)),$$

which satisfies the following properties:

(i) The first entry is an *index* taken from a well-ordered set $\mathscr{I}$ and each element of $\mathscr{I}$ occurs in precisely one $\mathscr{M}$-quintuple. The well-ordering of $\mathscr{I}$ satisfies the property that

$$\text{if } b_\infty(\omega) < b_\infty(\omega') \qquad \text{then } \iota(\omega) < \iota(\omega'). \tag{2.3.1}$$

(ii) The second entry is a bounded word sequence and each of those word sequences occurs in precisely one $\mathscr{M}$-quintuple.

(iii) $C(\omega)$ is the set of all closed $\mathbb{P}(\omega)$-intervals which occur as a coincidence pattern between our word sequence ω and any word sequence ω' with $\iota(\omega') < \iota(\omega)$.

(iv) Based on having put up the list of all coincidence patterns of $\mathbb{P}(\omega)$-intervals in (iii) we now define equivalence classes on our set $\mathbb{P}(\omega)$ of position sequences as follows: Regard $a < b \in \mathbb{P}(\omega)$ as equivalent, if there exists a finite cover of the interval $[a, b]$ with such intervals of position sequences that have just been listed in $C(\omega)$. The corresponding equivalence classes turn to be (closed, open, or semi-open) $\mathbb{P}(\omega)$-intervals, and we denote the set of all those intervals by $E(\omega)$ (cf. Lemma 2.10).

(v) $F(\omega)$ is a subset of position sequences of $\mathbb{P}(\omega)$ which is chosen so that $F(\omega)$ contains precisely one representative for each equivalence class listed in $E(\omega)$, and such that $0^{\mathbb{N}}$ is chosen for representing the equivalence class containing $0^{\mathbb{N}}$.

Note that Formula (1) gives the only requirement for $\mathscr{M}$ which is not merely either a definition, an elementary construction, or an immediate application of the well-ordering theorem ($\mathscr{I}$ is bijectively associated to H). In order to achieve this compatibility between our well-ordering indices ι and the b_∞-indices as defined in 0.2, proceed as follows:

The b_∞-function is well defined on the set of all word sequences, and for the set of bounded word sequences it only takes on finite values. Hence b_∞ can be used for subdividing $\mathscr{I}$ into countably many disjoint classes, the ith class being defined to consist of those word sequences where the b_∞-index is i. Now apply the well-ordering theorem to each of these classes separately, which gives some order on each of these classes. Then combine these orderings lexicographically with the natural well-ordering of the set of all possible b_∞-values and this gives the desired order of H which satisfies Formula (1).

2.4 *Remark.* Henceforth in this paper we think of $\mathscr{M}$ as our "main list"; i.e., we think of $\mathscr{M}$ as an infinite well-ordered list where all word sequences of the corresponding group are entered and where each of these entries is accompanied by the list of corresponding coincidence patterns, equivalence classes, and their representatives. According to this way of thinking of $\mathscr{M}$ we will sometimes use expressions like "ω' is listed before ω" instead of "$\iota(\omega') < \iota(\omega)$," and so on. The information contained in such a list will enable us in 2.7 to immediately define a candidate for a free basis of H.

2.5 Lemma. *For all entries of the main list $\mathscr{M}$ of a group of bounded word sequences, any internal coincidence pattern is an external one.*

Proof. For $a < b \le c < d \in \mathbb{P}(\omega)$ let $[c, d]$ be an internal coincidence pattern, i.e., $\omega_{|[a,b]} = \omega_{|[c,d]}$. Then $\eta = \omega_{|[a,b]}$ can also be regarded as a description for some word sequence η which apart from its occurrence as a substring of ω must be at some place directly entered into the main list. Note that whichever of our two entries ω and η comes first, it induces corresponding coincidence patterns at the other one. Hence proving that η comes before ω, i.e., that $\iota(\eta) < \iota(\omega)$, implies that $\mathbb{P}(\omega)$-intervals $[a, b]$ and $[c, d]$ are external coincidence patterns. Since we want to use 2.3(1) as the main argument in the corresponding proof, we have to count the letters in our word sequences η and ω. Because of $\eta = \omega_{|[a,b]} = \omega_{|[c,d]}$ we get that each letter which is contained in η has to have at least twice as many of occurrences in ω. This guarantees for finite b_∞-indices that $b_\infty(\eta) < b_\infty(\omega)$, and thus $\iota(\eta) < \iota(\omega)$ by 2.3(1). As explained before, this is the desired result. ∎

2.6. *Remark*. Note that the above Lemma 2.5 will be essentially quoted in the proof of Proposition 2.21. This should be understood as the crucial place in this paper, where we use the boundedness of our word sequences. It is this boundedness that guarantees that the b_∞-index is finite and that enables us to prove in the above Lemma 2.5 that we need not worry about internal coincidence patterns, cf. concluding Remarks 2.26 and 2.27.

2.7 Definition of a candidate $\mathscr{B}_1$ for a free basis of the group H of bounded word sequences. We obtain such a fixed candidate $\mathscr{B}_1$ by fixing a well-ordering on H and then letting

$$\mathscr{B}_1 := \left\{ \omega_{|[0^{\mathbb{N}}, p]} \mid \omega \in H, \, p \in F(\omega), \, p \neq 0^{\mathbb{N}} \right\}. \tag{2.7.1}$$

In words: Each entry of the main list contributes a certain number (it may be zero, one, or more than one) of substrings to our candidate for a free basis, namely all substrings which are bounded on the one hand by the initial position sequence of the corresponding entry, and on the other hand by those position sequences that have been chosen to be representatives of the corresponding equivalence classes.

2.8 *Remark and definition.* When selecting $\mathscr{B}_1$ in 2.7, each entry $\omega_{|[0^{\mathbb{N}}, p]}$ in this basis is naturally associated with an entry in the main list, and hence with all the accompanying entries of the main list such as the well-ordering index, the coincidence pattern, and equivalence classes of ω. This additional information is not contained in the set $\mathscr{B}_1$ if only regarded as a set of word sequences, but it is needed in the course of the proof that $\mathscr{B}_1$ is actually a free basis. For that purpose we now introduce the following set besides $\mathscr{B}_1$:

$$\mathscr{B} := \left\{ \left(\omega_{|[0^{\mathbb{N}}, p]}, \omega, p \right) \mid \omega \in H, \, p \in F(\omega), \, p \neq 0^{\mathbb{N}} \right\}.$$

To help understand the relation between both sets we point out that $\mathscr{B}_1$ is the set of the first components of the $\mathscr{B}$-triples.

2.9 *Remark*. Having defined our (hopeful) candidate for a free basis of H in 2.7/2.8, in principle two facts remain to be proved: namely, at first, that we have chosen enough elements to represent any word sequence, and, second, that we have chosen so few elements that we cannot represent the empty word sequence in a non-reduced way. We are heading at proving these two facts in 2.13 and 2.25 by first investigating the nature of the equivalence classes as listed in $E(\omega)$.

2.10 LEMMA. *Let A be an equivalence class of the position sequences of a word sequence ω in the sense of Definition* 2.3(iv). *Then A can be represented by a closed*, *open*, *or semi-open interval*.

Proof. It is immediate from the Definition 2.3(iv) that if $a, b \in A$, each position sequence c with $a \leq c \leq b$ has to be in A, too. By using 1.21 it follows from the topology of $\mathbb{R}$ that A can be represented as some interval. In this context note that $\bigcup_{i=1}^{\infty}[\frac{1}{i+1}, \frac{1}{i}]$ is an infinite covering of $(0,1]$ such that an interval spanned by any two points of $(0,1]$ can be covered by finitely many of the covering sets. This explains why equivalence classes, other than the generating intervals coming from coincidence patterns, can be open or semi-open, also.

2.11 CONVENTION FOR OUR NOTATION. Let $a \in \mathbb{P}(\omega)$. Then $E(\omega, a)$ denotes the equivalence class as listed in $E(\omega)$ to which a belongs (cf. 2.3(iv)). $\rho(a)$ denotes the corresponding representative of that equivalence class as listed in $F(\omega)$ (cf. 2.3(v)), $s(a)$ the lower bound of this equivalence class, and $t(a)$ the upper bound. $s(a)$ and $t(a)$ may or may not belong to $E(\omega, a)$ by 2.10, but all position sequences p with $s(a) < p < t(a)$ do. More precisely, $E(\omega, a)$ is either the closed interval or the open interval or one of the two semi-open intervals bounded by $s(a)$ and by $t(a)$.

2.12 *On the method for proving that our candidate $\mathscr{B}$ as selected in* 2.7(1)/2.8 *is at least a generating system*. In what follows we give an iterative process of replacing an arbitrary given word sequence by a product of word sequences. Throughout the whole iterative process each factor will be defined as some substring of some entry in the corresponding main list, making implicit use of the fact that $\mathscr{B}$ is defined as a set of triples without explicitly referring to the triple notation. The process only terminates when a decomposition is found such that each of the factors belongs to $\mathscr{B}$. We give below the rules for replacing those factors of product decompositions not belonging to $\mathscr{B}$ by new products. Each of these rules will evidently give a correct replacement; the critical point is to make sure that even for arbitrary elements this process terminates after finitely many steps. However, this will follow in 2.13, and in order to prove it we will explicitly use the fact that all word sequences of consideration are bounded. Doing this will also help to understand why one cannot deduce from these methods that the entire group, i.e., the fundamental group of the Hawaiian Earrings, is free—and in fact by other methods this group is known to be not free (cf. [dSm, Sec. 3; Zas, 4.8]). The corresponding process comprises the following steps:

(i) Initial step: Given an arbitrary word sequence $\omega \in H$, then find this ω in our main list and use the one-factor representation $\omega_{|[0^{\mathbb{N}}, \ell^{\mathbb{N}}]}$ to represent ω.

(ii) If $\omega_{|[a,b]}$ is one factor in the course of this process and if a and b belong to different equivalence classes of $E(\omega)$, then find the representatives $\rho(a)$ and $\rho(b)$ as selected for $F(\omega)$ and replace $\omega_{|[a,b]}$ by $\omega_{|[a,\rho(a)]} \cdot \omega_{|[\rho(a),0^{\mathbb{N}}]} \cdot \omega_{|[0^{\mathbb{N}},\rho(b)]} \cdot \omega_{|[\rho(b),b]}$ omitting those factors which are the empty word sequence according to 1.22.

(iii) If $\omega_{|[a,b]}$ occurs as a factor where a and b (without loss of generality $a < b$) belong to the same $E(\omega)$-interval but are not contained in one coincidence pattern of $C(\omega)$, choose finitely many position sequences a_i such that $a = a_0 < a_1 < a_2 < \cdots < a_k = b$ and such that each $[a_{i-1}, a_i]$ is covered by one interval of position sequences registered in $C(\omega)$. Then replace $\omega_{|[a,b]}$ by $\prod_{i=1}^{k} \omega_{|[a_{i-1},a_i]}$.

(iv) If $\omega_{|[a,b]}$ occurs as a factor in the iterative process such that $[a,b]$ belongs to an external coincidence pattern according to 2.1, then replace the factor $\omega_{|[a,b]}$ by the corresponding substring of some word sequence ω' that contains $\omega_{|[a,b]}$ as a substring and was previously listed in our main list.

2.13 Lemma. *The candidate $\mathscr{B}_1$ for a free basis of H as selected in* 2.7(1) *is a generating system*:

The proof is based on the fact that Process 2.12 cannot run endlessly. In this context note that the factors $\omega_{|[0^{\mathbb{N}},\rho(b)]}$ and $\omega_{|[\rho(a),0^{\mathbb{N}}]}$ as introduced in Step 2.12(ii) are by definition members or inverses of members of $\mathscr{B}_1$, and hence these factors need no further replacement in the course of this process. The splitting up of a word sequence as defined in 2.12(iii) cannot be applied repeatedly to some factor without transformations used by one of the other steps. Further note that in the course of transforming one factor one cannot apply 2.12(iv) infinitely many times since each application of this process is associated with some skipping to a smaller entry in the well-ordering of our main list. And a well-ordering does not permit the construction of an infinite descending chain. Hence Process 2.12 terminates after finitely many steps. In its termination state it gives a product decomposition of ω by finitely many $\mathscr{B}_1$-elements and their inverses, proving that $\mathscr{B}_1$ is a generating system.

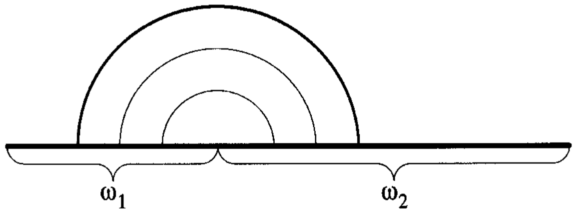
2.14 *On the proof that $\mathscr{B}_1$ is free*. In order to prove the second claim of Remark 2.9 we will have to consider a product of $\mathscr{B}_1$-elements which despite being "reduced" is assumed to be "completely cancellable." In this context *reduced* means that any two adjacent factors cannot comprise some $\mathscr{B}_1$-element and its inverse. On the other hand, *completely cancellable* refers to what happens when we concatenate the corresponding word sequences, namely that the resulting one can be cancelled till it is empty.

The role of this paragraph is to explain some of the conventions and assumptions that we can require without loss of generality, and to outline the argument of the subsequent proof, the essential technical arguments of which will be given in Propositions 2.18–2.25: Roughly speaking, the proof will be based on showing that certain substrings cannot cancel in such a product. These domains will in particular be found in the *maximal factors*. In this context "maximal" refers to the well-ordering index to which each $\mathscr{B}_1$-element and its inverse are naturally associated with by 2.8. Since the products of consideration are finite, for each such product there is a maximum of the associated well-ordering indices, and all of the corresponding factors are regarded as "maximal." Two maximal factors are called *adjacent* if there are not any other factors placed in between, while the attribute *neighboring* in this context only implies that none of the factors in between are maximal. In 2.17 we will give the precise definition of the corresponding domains which, as a rule of thumb, satisfy that they resist any cancelling if contained in the maximal factors.

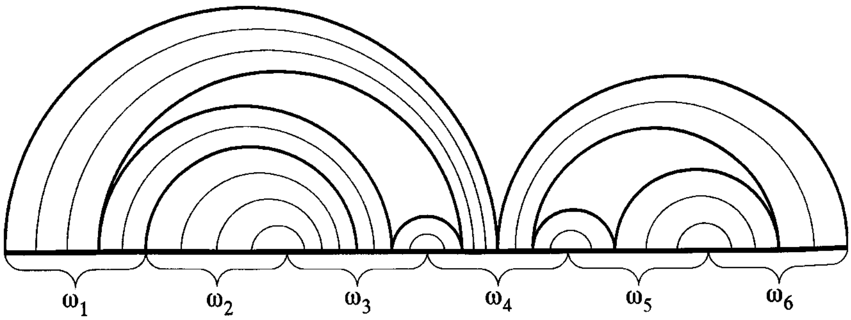
However, there is one exception from that rule, and due to this exception the domains defined in 2.17 may cancel in a product as described up to the moment. This is why we now need to perform an

($*$) *adaptation step*; i.e., we slightly change our point of view as to what word sequences are considered as "factors" of our product, and all future references and definitions (including 2.17) refer to the adapted version of our product. Our adaptation step consists of executing the multiplication of any two such adajcent factors that are drawn from the same $\mathscr{M}$-entry such that the first of them is an inverse and the second is not. In formulae: If we have a subproduct of the type $\omega_{|[b,0^{\mathbb{N}}]} \cdot \omega_{|[0^{\mathbb{N}},c]}$, these two factors are replaced by $\omega_{|[b,c]}$ which is to be considered as one factor in our product henceforth and which we still regard as associated with the $\mathscr{M}$-entry ω and with its well-ordering index. Note that under no circumstances can we get that $\omega_{|[b,c]}$ is the empty word sequence, since this would imply that $b = c$ and hence that the product as originally given could not have been reduced. Further note that neither can the adaptation step be applied consecutively to any of our factors, nor is there any freedom of choosing between the right or the left adjacent factor for the adaptation.

2.15 *Remark*. Fig. 2 gives a geometric picture for how cancellation (cf. 1.13) could appear in the course of executing a longer product. Recall that by Lemma 1.16 certain aspects of these pictures, especially their essential finite combinatorics, are common to all of them. In this context, when using the phrase *cancellation pattern*, we mean the finite subdivision of the word sequence factors of a longer product and the scheme of the order the



(a)



(b)

FIG. 2. Belonging to Remark 2.15. These figures try to visualize the cancellations as they (may) take place in the course of computing a product between word sequences. (a) gives the general picture of the product between two word sequences if one takes into account that any of the domains may be empty; (b) gives an impression of what the cancellation pattern of a product with more than two factors may look like. Within these figures the horizontal bars represent word sequences, and the semicircles ("arches") are supposed to connect those (places of) letters within the word sequences that are cancelled against each other at some stage of our process. Of course, such figures can only contain finitely many of a possibly infinite number of arches. However, note that, even if the ideal pictures of both figures might well contain infinitely many arches, both of figures contain only finitely many classes of parallel arches. This is because the atomic picture (a) contains by Definition 1.11(1) only one class of parallel arches, and this finiteness condition sustains in the course of an iterated execution of a longer multiplication (cf. 1.16).

corresponding substrings are to be cancelled against each other. Note that the cancellation pattern for a product is in general not uniquely defined. In this context we have uniqueness for a product with two factors by considering greatest common initial word sequences, but the cancellation pattern of a longer product still might depend on how the product is parenthesized. By 1.19, what remains as a reduced word sequence after all cancellations have been performed is independent of the parenthesizing, but the intermediate steps may depend on it. Henceforth in our proof we will associate a fixed cancellation pattern with a product Π, and thus we will also define and refer to objects that depend on the precise cancellation pattern.

2.16 DEFINITION. Let Π be a product of word sequences, and use η_i to denote the ith factor of Π. Let $0^{\mathbb{N}} = a_0 < a_1 < a_2 < \cdots < a_n = \ell^{\mathbb{N}}$ be the subdivision of $\mathbb{P}(\eta_i) = [0^{\mathbb{N}}, \ell^{\mathbb{N}}]$ which gives the compatible segmentation of η_i in the sense of 2.15/1.16 for the cancellation pattern of Π. For $a \in \mathbb{P}(\eta_i) - \{a_0, \ldots, a_n\}$ we introduce the notation in order to describe the segment of η_i that contains the position sequence a,

$$\Sigma_i(a, \Pi) := \Sigma_i^+(a, \Pi) := \Sigma_i^-(a, \Pi) := [a_{j-1}, a_j],$$

where the index j is chosen such that $a_{j-1} < a < a_j$. For the remaining a-values, i.e., for those which coincide with one of the a_k, the domain $\Sigma_i(a, \Pi)$ is not defined; however, in this case we let $\Sigma_i^-(a_k, \Pi) := [a_{k-1}, a_k]$ and $\Sigma_i^+(a_k, \Pi) := [a_k, a_{k+1}]$.

2.17 DEFINITION OF THE DOMAINS $\mathscr{S}_i(\omega, \Pi)$ AND $\mathscr{T}_i(\omega, \Pi)$. Let $\eta_1 \cdot \cdots \cdot \eta_k$ be the product of word sequences as denoted by Π, and in the following consider in particular the factor $\eta_i = \omega_{|[a,b]}$. Here ω denotes the entry in the main list $\mathscr{M}$ with which our factor η_i is associated according to 2.8 (for notation cf. 2.11). For $a < b$ we let

$$\begin{aligned} \mathscr{S}_i(\omega, \Pi) &:= \begin{cases} \Sigma_i^-(t(a), \Pi) & \text{if } t(a) \notin E(a, \omega), \\ \Sigma_i^+(t(a), \Pi) & \text{if } t(a) \in E(a, \omega), \end{cases} \\ \mathscr{T}_i(\omega, \Pi) &:= \begin{cases} \Sigma_i^-(s(b), \Pi) & \text{if } s(b) \in E(b, \omega), \\ \Sigma_i^+(s(b), \Pi) & \text{if } s(b) \notin E(b, \omega). \end{cases} \end{aligned} \tag{2.17.1}$$

In case $b < a$ the $\mathscr{S}$- and $\mathscr{T}$-domains are defined similarly; the above formulae merely have to be adapted by replacing $t(a)$ by $s(a)$ and $s(b)$ by $t(b)$, and by putting "$(\cdots)^{-1}$" around all strings "$\Sigma_i^{\pm}(\ldots, \Pi)$." In this context note that we always want the $\mathscr{S}$- and $\mathscr{T}$-domains to be subsets of $\mathbb{P}(\omega)$, in contrast to the Σ-domains which by 2.16 have been defined as subsets of $\mathbb{P}(\eta_i)$. However, for the sake of keeping our formulae simple we did not introduce an extra notation for the natural embedding $\mathbb{P}(\eta_i) = \mathbb{P}(\omega_{|[a,b]}) \hookrightarrow \mathbb{P}(\omega)$ $(a < b)$ which one might have used in the above formula. In case $b < a$, where we only have a natural embedding $\mathbb{P}(\eta_i) = \mathbb{P}(\omega_{|[a,b]}) \hookrightarrow \mathbb{P}(\omega^{-1})$ we also need to perform an inversion in order to define our $\mathscr{S}$- and $\mathscr{T}$-domains as subsets of $\mathbb{P}(\omega)$ in this case, too. For this purpose we have to write "$(\ldots)^{-1}$." We also introduce notations for the bounding positions of our $\mathscr{S}$- and $\mathscr{T}$-domains, so that we can write either

$$\mathscr{S}_i(\omega,\Pi) = [\sigma_1,\sigma_2] \quad \text{or} \quad \mathscr{S}_i(\omega,\Pi) = [\sigma_2,\sigma_1] \quad \text{and}$$
$$\mathscr{T}_i(\omega,\Pi) = [\tau_1,\tau_2] \quad \text{or} \quad \mathscr{T}_i(\omega,\Pi) = [\tau_2,\tau_1]. \qquad (2.17.2)$$

In particular, we define σ_1, τ_1, σ_2, and τ_2 so that in the subsequent formula in each line one of the two inequality chains is satisfied:

$$\begin{aligned} &a \le \sigma_1 < \sigma_2 \le b \quad \text{or} \quad a \ge \sigma_1 > \sigma_2 \ge b,\\ &a \le \tau_1 < \tau_2 \le b \quad \text{or} \quad a \ge \tau_1 > \tau_2 \ge b. \end{aligned} \qquad (2.17.3)$$

Note that $\mathscr{S}_i(\omega,\Pi)$ and $\mathscr{T}_i(\omega,\Pi)$ might have a non-trivial intersection; however, the corresponding equivalence classes used for their definition in Formula (1), i.e, $E(a, \omega)$ and $E(b, \omega)$, must be disjoint in any case (even if their closures need not be). This is a consequence of the fact that by 2.14($*$)/2.7 our factor $\eta_i = \omega_{|[a,b]}$ has been defined as bounded by two different representatives of equivalence classes.

2.18 PROPOSITION. *In any adapted product $\eta_1 \cdot \cdots \cdot \eta_n$ with one single maximal factor $\eta_i = \omega_{|[a,b]}$ the domains $\mathscr{S}_i(\omega,\Pi)$ and $\mathscr{T}_i(\omega,\Pi)$ cannot cancel against any subdomains of other factors.*

Proof. We restrict our considerations, without loss of generality, to the $\mathscr{S}$-domain. Assume that $\mathscr{S}_i(\omega,\Pi)$ is cancelled in the course of the execution of the product Π, which means that the corresponding partner domain is some Σ-domain which must lie in some other factor $\eta_j = \omega'_{|[a',b']}$ with $\iota(\omega') < \iota(\omega)$. Hence the entire domain $\mathscr{S}_i(\omega,\Pi)$ is a coincidence pattern, and it has to be contained in one equivalence class that is listed in $E(\omega)$ (cf. 2.3(iv)). Note that $\mathscr{S}_i(\omega,\Pi)$ is always a closed interval. This contradicts to 2.17(1) which has defined a different relation between $\mathscr{S}_i(\omega,\Pi)$ and the corresponding equivalence class $E(a, \omega)$.

2.19 COROLLARY. *Any reduced completely cancellable product has more than one maximal factor. Let $\eta_i = \omega_{|[a,b]}$ and $\eta_j = \omega_{|[c,d]}$ denote two neighboring maximal factors, which occur in our product Π. Then there exist such factors where the $\mathscr{T}$-domain of the first, which will equivalently be denoted by $\omega_{|\mathscr{T}_i(\omega,\Pi)}$ or by $\omega_{|[\tau_1,\tau_2]}$, cancels against $\omega_{|\mathscr{S}_j(\omega,\Pi)} = \omega_{|[\sigma_1,\sigma_2]}$, i.e., the $\mathscr{S}$-domain of the second factor.*

Proof. Based on the same line of arguments as in the preceding Proposition 2.18 we get that $\mathscr{S}$- and $\mathscr{T}$-domains of maximal factors cannot cancel against substrings of non-maximal factors. For similar reasons they cannot cancel against inner substrings of $\omega_{|[c,\sigma_1]}$ and $\omega_{|[\tau_2,b]}$. The remaining claim of this corollary follows from searching the pattern (cf. 2.15 for an innermost place where such $\mathscr{S}$- and $\mathscr{T}$-domains are associated with each other). Note that $\mathscr{S}$- and $\mathscr{T}$-domains have been defined as a particular Σ-domains (cf. 2.16/2.17(1)), and hence they are not given the possibility for "partial cancellation." They can only cancel completely against each other, or they may cancel completely against other Σ-domains, and if both do not happen they remain uncancelled and appear in the word sequence describing the product. Since, on the other hand, our product was assumed to be completely cancellable, our cancellation pattern must contain $\mathscr{S}$- and $\mathscr{T}$-domains as claimed above. ∎

2.20 *Remark.* Recall that by 2.19/2.17 our Assumption 2.14 led to the hypothesis that $\mathscr{S}_j(\omega, \Pi)$ and $\mathscr{T}_i(\omega, \Pi)$ both are domains which are contained in $\mathbb{P}(\omega)$. *The following Propositions* 2.21–2.24 *may be understood as a part of the proof that Assumption* 2.14 *yields a contradiction, where each of these propositions is devoted to rule out a certain possibility of how* $\mathscr{S}_j(\omega, \Pi)$ *and* $\mathscr{T}_i(\omega, \Pi)$ *may be situated in* $\mathbb{P}(\omega)$ *with respect to each other.*

2.21 PROPOSITION. *Let the assumptions and notation be as in Corollary* 2.19. *Then* $\mathscr{S}_j(\omega, \Pi)$ *cannot be disjoint from* $\mathscr{T}_i(\omega, \Pi)$.

Proof. By assumption, $\omega_{|\mathscr{T}_i(\omega, \Pi)}$ cancels against $\omega_{|\mathscr{S}_j(\omega, \Pi)}$. Hence the corresponding intervals $\mathscr{T}_i(\omega, \Pi)$ and $\mathscr{S}_j(\omega, \Pi)$ mark internal coinciding patterns within ω. By 2.5 these intervals are external coinciding patterns as well, and then from the same argument as in 2.18 we obtain a contradiction. ∎

2.22 PROPOSITION. *Let the assumptions and notation be as in Corollary* 2.19. *Then* $\tau_1 \neq \sigma_1$ *or* $\tau_2 \neq \sigma_2$.

Proof. If we are assuming the contrary in this case, i.e., that $\sigma_1 = \tau_1$ and that $\tau_2 = \sigma_2$, then we are assuming that there exists a substring of ω, namely $\omega_{|[\sigma_1, \sigma_2]}$, which is inverse to itself. In particular, for each finite word $\alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_n}$ of $\omega_{|[\sigma_1, \sigma_2]}$ we have that $\alpha_{i_1} = \alpha_{i_n}^{-1}$, $\alpha_{i_2} = \alpha_{i_{n-1}}^{-1}, \ldots,$ and so on. This in particular implies that n is even, so that our word has a "middle position" splitting up this word into two halves such that the first of them is inverse to the second. These middle positions together give a position sequence which has the corresponding splitting up property for the entire word sequence $\omega_{|[\sigma_1, \sigma_2]}$. Having found such a position sequence gives that ω cannot be reduced—which contradicts our basic assumption for all word sequences of $\pi_1(Y) \supset H \ni \omega$ (cf. the first sentences of Section 2). ∎

2.23 PROPOSITION. *Let the assumptions and notation be as in Corollary* 2.19. *Then* $\mathscr{T}_i(\omega, \Pi)$ *cannot be partial overlapping with* $\mathscr{S}_j(\omega, \Pi)$.

Proof. Note that in case of a product where more than one factor is derived from the same entry ω in the main list we cannot expect compatibility between the different subdivisions as they are induced from these different factors according to 1.16 on $\mathbb{P}(\omega)$. Hence partial overlapping domains do not yield a contradiction by definition already. However, we will find one by discussing now how such domains inducing identical substrings of ω could be situated in $\mathbb{P}(\omega)$: Note that in any such case there exists a finer subdivision of the corresponding substrings, giving a number of pairs of identical finer substrings satisfying that the two members of each of the pairs are either situated in disjoint areas of $\mathbb{P}(\omega)$ or are inverse to itself. If the latter occurs, we have an immediate contradiction by 2.22; if only the first occurs, each of the finer substrings can be considered as an internal coincidence pattern, and since the covering by finitely many coincidence patterns is sufficient to conclude that the corresponding string has to be contained in one equivalence class only, we can use the same arguments like in 2.21 to come to a contradiction in this case also. ∎

2.24 Proposition. *Let the assumptions and notations be as in Corollary* 2.19. *In addition assume that $\tau_1 = \sigma_2$ and that $\tau_2 = \sigma_1$. Then the corresponding maximal factors are not in adjacent positions. In addition, the subproduct of those factors that are placed between our two maximal ones cancels completely.*

Proof. Consider $\eta_i \cdot \eta_{i+1} \cdot \cdots \cdot \eta_{j-1} \cdot \eta_j$, i.e., the subproduct of Π which is bounded by the corresponding maximal factors $\eta_i = \omega_{|[a,b]}$ and $\eta_j = \omega_{|[c,d]}$. Since by assumption the factors in our product Π are defined as $\mathscr{B}_1$-elements and their inverses, they can only be substrings of $\mathscr{M}$-entries bounded by representatives of equivalence classes. Hence our assumption that $\mathscr{T}_i(\omega, \Pi) = \mathscr{S}_j(\omega, \Pi)$ and the fact that these $\mathscr{S}$- and $\mathscr{T}$-domains are defined in correspondence to the pattern of equivalence classes of $E(\omega)$ give that $b = c$. In addition we get from this coincidence of the $\mathscr{S}_j$- and the $\mathscr{T}_i$-domain and of the corresponding equivalence classes that b can only be the maximum or the minimum of the set $\{a, b, d\}$. Now assume that our subproduct $\eta_{i+1} \cdot \cdots \cdot \eta_{j-1}$ is void, so that η_i and η_j are in adjacent positions within Π. Then the corresponding subproduct is either of type $\eta_i \cdot \eta_j = \omega_{|[a,0^{\mathbb{N}}]} \cdot \omega_{|[0^{\mathbb{N}},d]}$ or it is of type $\eta_i \cdot \eta_j = \omega_{|[a,b]} \cdot \omega_{|[b,d]}$ with $b \neq 0^{\mathbb{N}}$. Both cases give a contradiction, since in the first case the corresponding subproduct should have been adapted to one factor $\omega_{|[a,d]}$ in 2.14, and in the second case we can conclude that the original adjacent factors, as they had looked before our adaptation step, must have been $\omega_{|[0^{\mathbb{N}},b]}$ and $\omega_{|[b,0^{\mathbb{N}}]}$, contradicting the fact that we are considering a reduced product in 2.9/2.14. This contradiction shows that $\eta_{i+1} \cdot \eta_{i+2} \cdot \cdots \cdot \eta_{j-1}$ is a non-void product and proves the first claim of this proposition.

Let η be the word sequence which is obtained from computing the product $\eta_{i+1} \cdot \eta_{i+2} \cdot \cdots \cdot \eta_{j-1}$ in such a way that we are performing those cancellations only which according to the given cancellation pattern of the original product Π are taking place within these factors. Now consider the product $\eta_i \cdot \eta \cdot \eta_j$ with $\eta_i = \omega_{|[a,b]}$ and $\eta_j = \omega_{|[b,d]}$ as above. Since our product $\eta_1 \cdot \cdots \cdot \eta_k$ was assumed to cancel completely and since in particular the $\mathscr{S}$-domain of η_i is assumed to cancel against the $\mathscr{T}$-domain of η_j, in the course of the cancellation process the first half of η must cancel against the end of the preceding η_i-factor while the second half must cancel against the beginning of the subsequent η_j-factor. Let $[p,b]$ and $[b,q]$ denote the corresponding $\mathbb{P}(\omega)$-intervals which cancel that way. As can be concluded easily from the cancellation pattern, these intervals have to be placed next to $\mathscr{T}_i(\omega, \Pi)$ and $\mathscr{S}_j(\omega, \Pi)$, respectively. Since the latter intervals have been assumed to coincide, we conclude that $p = \tau_2 = \sigma_1 = q$. Thus we get that the first half of η is $(\omega_{|[p,b]})^{-1}$ and that the second half of η is $(\omega_{|[b,p]})^{-1}$. Hence η can be cancelled up to the empty word sequence. That way we finally see that $\eta_{i+1} \cdot \eta_{i+2} \cdot \cdots \cdot \eta_{j-1}$ can be cancelled (via η as a intermediary step) up to the empty word sequence, as it has been the second claim of this proposition. ∎

2.25 LEMMA. *Any (finite) product of $\mathscr{B}_1$-elements and their inverses cannot give the empty word sequence, provided it does not contain adjacent factors which are inverse to each other in the sense that one is of type $\omega_{|[a,b]}$ and the other then is $\omega_{|[b,a]}$ (cf. 2.8).*

Proof. This proof is a transfinite induction according to the well-ordering of our candidate $\mathscr{B}_1$ for a free basis; or, alternatively, it can be understood as searching for the lowest well-ordering index ι_0 which satisfies that there exists a finite reduced product whose maximal well-ordering index is ι_0, but which is completely cancellable on the level of word sequences (cf. 2.14). In principle, the arguments for the zero-step and the inductive step of this induction have been given in the preceding Propositions 2.18–2.24: Assume that we have found such a product. Without loss of generality we can think of this product as being adapted as in 2.14($*$). By 2.18 the maximal well-ordering index of such a product has to occur more than once, and by 2.19 one can in particular find two neighboring occurrences where the $\mathscr{T}$-domain of the first cancels against the $\mathscr{S}$-domain of the second. Mark the corresponding $\mathscr{S}$- and $\mathscr{T}$-domains within $\mathbb{P}(\omega)$, where ω is the word sequence associated with the maximal well-ordering index ι_0. These domains can only either be disjoint, or overlapping, or coinciding, where the latter case is subdivided into two cases by making a distinction whether the domains coincide with reversed or with matching orientation. It was discussed in Propositions 2.21–2.23

that the first three of these four cases give a contradiction. In the remaining case by Proposition 2.24 we find a subproduct that is reduced but completely cancellable such that all of its factors have a lower well-ordering index. Since this contradicts our induction hypothesis, the inductive step is complete.

The zero-step of this induction follows by the same arguments, since in the zero-step we only have to consider a product of word sequences drawn from the same first entry in the list $\mathscr{M}$, and hence there cannot be a subproduct of word sequences with lower well-ordering indices as found in 2.24. Thus in the case of the zero-step we directly get a contradiction from 2.24. ∎

2.26 *Remark on the role of the condition "bounded."* If we dropped the assumption "bounded," we would have to allow that ∞ is a legitimate value for b_∞; however, since $\mathbb{N} \cup \{\infty\}$ is a well-ordered set, 2.3(1) still could have been fulfilled. Of course, for $b_\infty(\omega) = \infty$, we might have subpatterns of ω with the same b_∞-index. For that reason we cannot get around internal coincidence patterns any longer, and we have to extend our processes 2.3/2.12 so as to cope with them also. At first sight this seems perfectly doable: In 2.3(iii) we would have in addition to require that each internal coincidence pattern of a word sequence is included in $C(\omega)$ apart from its first occurrence, and in 2.12 we would have to add a fifth step requiring that a substring covered by such an internal coincidence pattern is replaced by the corresponding substring covered by the first occurrence of this pattern. However, performing such a "Step 2.12(v)" does not reduce the well-ordering index, and hence the arguments of 2.13 could in this case only give that 2.12 is a process which either terminates after finitely many steps, or which is trapped in an infinite loop where one has to apply 2.12(v) again and again to the same entry of the main list. Such situations can be constructed: Use as example the subgroup generated by all substrings of the word sequences [Zas, (4.2)], and assume that the corresponding well-ordering satisfies that the generating word sequence ω or ω^{-1} is the lowermost entry of the class which has ∞ as its b_∞-index. Then the run of the process 2.3/2.12 can in many cases explicitly be traced and loops with 2.12(v) can be found. From such examples one can also see that with the methods as developed in this paper there is no chance of being able to give the result that the entire group $\pi_1(Y)$ might be free (incidentally this result has been proven to be wrong, cf., e.g., [dSm, Sec. 3]).

2.27 *Concluding remarks.* In principle, this paper may also be understood as a piece of work which was made to help understand the obstructions that prevent the fundamental group of such a graph-like space like the Hawaiian Earrings Y from being free. One might find this fact hard to understand, in particular because $\pi_1(Y)$ can be described by some calculus of infinite words which does not involve more equivalence relations than known from the calculus of finitely generated free groups. By contributing our basis selection mechanism (2.3/2.7) we show that the obstruction that prevents $\pi_1(Y)$ from being free is probably not related to the fact that there is not any obvious candidate for a free basis within the group of word sequences. Moreover, by analyzing the situation in which our basis selection mechanism fails to generate a free basis, our attention is drawn to the phenomenon of word sequences having infinitely many iterated repeating subpatterns (cf. 2.26). This is a phenomenon which cannot be found in finitely generated free groups.

REFERENCES

[C-C] J. W. Cannon and G. R. Conner, "The Combinatorial Structure of the Hawaiian Earring Group," preprint, Brigham Young University, Provo, UT, 1997.

[Eda] Katsuya Eda, "Free Subgroups of the Fundamental Group of the Hawaiian Earring," preprint, Waseda University, Tokyo, 1998.

[FS] U. Felgner and K. Schultz, Algebraische Konsequenzen des Determiniertheitsaxioms, *Arch Math*. **42** (1984), 557–563.

[Gr1] H. B. Griffiths, The fundamental group of two spaces with a common point, *Quart. J. Math. Oxford* (2) **5** (1954), 175–190; correction: *ibid*. **6** (1955), 154–155.

[Gr2] H. B. Griffiths, Infinite products of semi-groups and local connectivity, *Proc. London Math. Soc*. (3) **6** (1956), 455–480.

[Higm] Graham Higman, Unrestricted free product, and varieties of topological groups, *J. London Math. Soc*. (2) **27** (1952), 73–88.

[MM] J. Morgan and I. Morrision, A van-Kampen-theorem for weak joins, *Proc. London Math. Soc*. (3) **53** (1986), 562–576.

[Noe] G. Nöbeling, Verallgemeinerung eines Satzes von Herrn E. Specker, *Invent. Math*. **6** (1968), 41–55.

[dSm] B. de Smit, The fundamental group of the Hawaiian Earring is not free, *Internat. J. Algebra Comput*. **2**, No. 1 (1992), 33–37.

[Sp] Ernst Specker, Additive Gruppen von Folgen ganzer Zahlen, *Portugaliae Math*. **9**, No. 3 (1950), 131–140.

[Zas] A. Zastrow, Construction of an infinitely generated group that is not a free product of surface groups and abelian groups, but which acts freely on an $\mathbb{R}$-tree, *Proc. Roy. Soc. Edinburg Sect. A* **128** (1998), 433–445.